Prime Numbers, Pythagorean Triples and Goldbach's Conjecture

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Content

§1 A Roadmap for generating Pythagorean Triples.

To find the solutions, represented by prime numbers, of

$$a^2 + b^2 = c^2, \quad \textbf{(1.1)}$$

where a, b en c are positive integer numbers, $\{a, b, c \in \mathbb{N} | a, b, c > 0\}$, is a standard procedure.

Instructive reading on Pythagorean Triples can be found, e.g., in www.en.wikipedia.org .

a, b en c constitute a so-called Pythagorean Triple. In general a is even and b and c are odd. The approach we choose is based on The Fundamental Theorem of Arithmetic.

The Theorem: any integer greater than 1 is either a prime number, or can be written as a unique product of prime numbers.

We write a as a product of prime numbers, p_s:

$$a := 2^l 3^{n_1} 5^{n_2} 7^{n_3} \dots \dots p_s^{n_s} \dots \dots\dots , \quad \textbf{(1.2)}$$

with $n_1, n_2, \dots \dots, n_s \dots. \ , \geq 0$.

So $a = 2^l \prod_{s=1}^{\infty} p_s^{n_s}$.

With $\{n_s \in \mathbb{N} \cup 0\}$ and $\{l \in \mathbb{N} | l > 0\}$.

p_s belongs to the subset of odd prime numbers $\mathbb{P}$: $\{p_s \in \mathbb{P} | \ p_s$ odd$\}$ and $\mathbb{P} \subset \mathbb{N}$.

Furthermore: $p_{s+1} > p_s$.

b en c can be expressed in a, with **(1.1)** , :

$$(c + b)(c - b) = a^2. \quad \textbf{(1.3)}$$

$c + b$ and $c - b$, $\{c \pm b \in \mathbb{N} | c \pm b > 0$ and even$\}$.

Euclid proofed the existence of an infinite number of Pythagorean triples.

$c+b$ and $c-b$ are integer factors of a^2. For example: with $a=4$, $c+b=8$ and $c-b=2$.
With Eq. (**1.3**) we have $c+b<a^2$, since c$-b>1$.
Also $c-b<a^2$, since $c+b>1$.
With $c+b$ and $c-b$ to be even positive integers, $c+b$ and $c-b$ are integer factors of a^2 indeed. So $c+b=\frac{a^2}{c-b}$, and $c-b=\frac{a^2}{c+b}$.
For convenience we write $a=2^l\prod_{s=1}^{\infty}p_s^{n_s}=2^lP_iP_j$, **(1.4)**
where P_i and P_j are products of powers of odd prime numbers. P_i and P_j being co-prime: $P_i\cap P_j=\emptyset$ and $\{P_i, P_j \subseteq \prod_{s=1}^{\infty}p_s^{n_s}\}$.
For $n_s=0$: $P_i=1$ and $P_j=1$.
For $n_s\neq 0$: a possible combination of $\{P_i, P_j\}$ is:
$P_i=\prod_{s=1}^{\infty}p_s^{n_s}$ and $P_j=1$,
and
$P_j=\prod_{s=1}^{\infty}p_s^{n_s}$ and $P_i=1$.
Furthermore: $P_i<P_j$ or $P_j<P_i$ **.**

The lowest possible triple is obtained for $n_s=0$ and $l=2$: (4, 3, 5).
For any combination of b and c, P_i en P_j can be found from different combinations of products of powers of prime numbers given in Eq. (**1.2**).
In general, with **(1.4)**, we have:
$(c+b)(c-b)=a^2=2^{2l}P_i^2P_j^2$. **(1.5)**
Since b and c are co-prime note the above representation of a in **(1.4)** is not just for convenience. With **(1.5)** , $c+b$ and $c-b$ can be represented respectively:
$c+b=2^{2l-k}P_i^2$, **(1.6)**
$c-b=2^kP_j^2$. **(1.7)**
For c and b we have:
$c=2^{2l-k-1}P_i^2+2^{k-1}P_j^2$, **(1.8)**
$b=2^{2l-k-1}P_i^2-2^{k-1}P_j^2$. **(1.9)**
$\{k\in\mathbb{N}|0<k<2l\}$.
Keep in mind: $c>b$ and c, b odd. Consequently $c-b>1$ and $(c+b)<2^{2l}P_i^2P_j^2$.
To make c and b co-prime indeed, we need to do something additional, unless:
$k-1=0$ or $2l-k-1=0$, **(1.10)**
we do not obtain a primitive Triple.
With $k=1$, $2l-k-1$ has to be larger than 0, or $2l-1-1>0\rightarrow l>1$,
and with
$2l-k-1=0\rightarrow k=2l-1\rightarrow k-1=2l-2$,
hence $l>1$.
So, these two constraints lead to the following condition: $l>1$. Since for $l=1$, and the

aforementioned constraints, the right-hand sides of Eqs. **(1.8)** and **(1.9)** represent the sum of two odd integers and become consequently even.
a , in the Triples you are looking for, is at least an integer that can be divided by 4.
Again: for $l = 2, k = 1$ we find the Triple (4, 3,5).
We also need the constraint: $b > 0$. With **(1.9)** we obtain:

$$2^{2l-k-1}P_i^2 > 2^{k-1}P_j^2 \;\rightarrow \frac{P_i}{P_j} > \frac{2^k}{2^l} \;. \qquad \textbf{(1.11)}$$

To summarize:

$$a = 2^l \prod_{s=1}^{\infty} p_s^{n_s} \equiv 2^l P_i P_j \,,$$

and
Table Roadmap:

k	P_i	P_j	$\frac{P_i}{P_j}$	$\frac{2^k}{2^l}$	Triple	$+\rightarrow c$ $-\rightarrow b$
1	$\prod_s p_s^{n_s}$	1	$\prod_s p_s^{n_s}$	$\frac{1}{2^{l-1}}$	yes	$2^{2l-2}P_i^2 \pm 1$
1	$\prod_{i} p_i^{n_i}$, $i \neq j$	$\prod_{j} p_j^{n_j}$, $j \neq i$	$\frac{\prod_i p_i^{n_i}}{\prod_j p_j^{n_j}}$	$\frac{1}{2^{l-1}}$	?	$2^{2l-2}P_i^2 \pm P_j^2$
1	1	$\prod_s p_s^{n_s}$	$\frac{1}{\prod_s p_s^{n_s}}$	$\frac{1}{2^{l-1}}$	?	$2^{2l-2} \pm P_j^2$
1	1	1	1	$\frac{1}{2^{l-1}}$	yes	$2^{2l-2} \pm 1$
$2l-1$	$\prod_s p_s^{n_s}$	1	$\prod_s p_s^{n_s}$	2^{l-1}	?	$P_i^2 \pm 2^{2l-2}$
$2l-1$	$\prod_{i} p_i^{n_i}$, $i \neq j$	$\prod_{j} p_j^{n_j}$, $j \neq i$	$\frac{\prod_i p_i^{n_i}}{\prod_j p_j^{n_j}}$	2^{l-1}	?	$P_i^2 \pm 2^{2l-2}P_j^2$
$2l-1$	1	$\prod_s p_s^{n_s}$	$\frac{1}{\prod_s p_s^{n_s}}$	2^{l-1}	no	
$2l-1$	1	1	1	2^{l-1}	no	

Note: the minimum value of $l = 2$.

§2 Pythagorean Triples.

Let us start with $l = 2, n_s = 0$ and with **(1.10)** $k = 1$ or $k = 3$.
- $k = 1$.
We have $a = 4$,

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 1 > \frac{2^k}{2^l} = \frac{1}{2}$.

With **(1.8)** and **(1.9):**

$c = 4 + 1 = 5$,

$b = 4 - 1 = 3$.

We obtain the triple we are familiar with: (4, 3, 5) .

- $k = 3$.

There is no triple. The triple is not allowed due the constraint **(1.11) .**

Now for the next triple(s): $l = 2, n_s = 1, s = 1$, $p_s = 3$ and with **(1.10)** $k = 1$ or $k = 3$.

- $k = 1$.

We have the smallest prime: 3.

$P_i = 3$ and $P_j = 1$.

Then $a = 4 \times 3 = 12$,

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 3 > \frac{2^k}{2^l} = \frac{1}{2}$.

from **(1.8)** and **(1.9)** we find:

$c = 4 \times 9 + 1 = 37$,

$b = 4 \times 9 - 1 = 35$.

The triple is: (12, 35, 37).

- $k = 3$.

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 3 > \frac{2^k}{2^l} = 2$.

So we expect another triple.

With **(1.8)** and **(1.9):**

$c = 9 + 4 = 13$,

$b = 9 - 4 = 5$.

The triple is (12, 5, 13).

So we have two triples for $a = 4 \times 3 = 12$.

However in the algorithm you need to find out about $P_i = 1$ and $P_j = 3$.

The constraint **(1.11)** forbids another triple.

Let us try another one.

Now for the next triple(s): $l = 2, n_s = 1, s = 1$, $p_s = 5$ and with **(1.10)** $k = 1$ or $k = 3$.

- $k = 1$.

We use the second smallest prime: 5 .

$P_i = 5$ and $P_j = 1$.

Then $a = 4 \times 5 = 20$,

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 5 > \frac{2^k}{2^l} = \frac{1}{2}$.

from **(1.8)** and **(1.9)** we find:

$c = 4 \times 25 + 1 = 101$,

$b = 4 \times 25 - 1 = 99.$

- $k = 3.$

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 5 > \frac{2^k}{2^l} = 2$.

So we expect another triple.

With **(1.8)** and **(1.9):**

$c = 25 + 4 = 29$,

$b = 25 - 4 = 21.$

The triple is (20, 21, 29).

We have two triples for $a = 20$.

For a particular value of a there are more than 2 non-trivial or primitive Pythagorean Triples.

An example summarized in the Table below:

$a = 60(= 2^2.3.5)$ produces four Pythagorean Triples :

(60, 11, 61) , $(60, 91, 109)$, $(60, 221, 229)$ and $(60, 899, 901)$.

So, with $a = 2^l \prod_{s=1}^{2} p_s^{n_s}$, $n_s = 1, s = 2,$ $p_1 = 3$, $p_2 = 5$ and $l = 2$:

k	P_i	P_j	P_i/P_j	$2^k/2^l$	triple
1	3× 5	1	15	1/2	Yes: 60, 899, 901
1	3	5	3/5	1/2	Yes: 60, 11, 61
1	5	3	5/3	1/2	Yes: 60, 109, 91
1	1	3× 5	1/15	1/2	no
3	3 × 5	1	15	2	Yes: 60, 221, 229
3	3	5	3/5	2	no
3	5	3	5/3	2	no
3	1	3× 5	1/15	2	no
					Four triples

Note:

- given $a = 180(= 2^2.3^2.5)$, so $n_1 = 2$ you will find again four triples.
- given $a = 2^2.3.5.7$, you will find eight triples.
- given $a = 2^2.3.5.7.11$, you will find sixteen triples.

§3 Triples and Goldbach's Conjecture Proved?

In the analysis above, I choose a to be even. Let's set a to be odd:

$a = P_i P_j$.

Similar to the above analysis we find:

$$c = \frac{P_i^2 + P_j^2}{2} , \qquad \textbf{(2.1)}$$

and

$$b = \frac{P_i^2 - P_j^2}{2} . \qquad \textbf{(2.2)}$$

The question is: c odd or b ? From the above expressions the conclusion could be:
b is "more even" than c , since

$$b = \frac{(P_i+P_j)(P_i-P_j)}{2} . \quad \textbf{(2.3)}$$

With

$$P_i^2 = 2b + P_j^2$$

we have

$$c = b + P_j^2 .$$

Since b is even, c is odd. Suppose c to be even, consequently b has to be odd. That contradicts $b = \frac{(P_i+P_j)(P_i-P_j)}{2}$.

Hence , I conclude c can never be even. In addition c the largest number in a triple. In trivial triple, $b = 0$, we have $c = a$.

In a non-trivial triple the largest number is never even.

Apart from that, by choosing a to be odd the algorithm to find the triple becomes more simple. The only constraint to be taken care off is:

$$\frac{P_i^2}{P_j^2} > 1 , \quad \textbf{(2.4)}$$

or with **(2.1)**:

$$c = \frac{P_i^2+P_j^2}{2} > P_iP_j = a .$$

Consequently:

$$P_i^2 - 2P_iP_j + P_j^2 > 0 \ \rightarrow P_i - P_j > 0 .$$

Start with $a = P_iP_j = 3$.
So $P_i = 3$ and $P_j = 1$, (the powers of the prime numbers in P_j are all zero, and we have the triple $(5,4,3)$.

There is more.
Let's look at **(2.3)** . Rearranging we obtain:

$$\frac{2b}{(P_i-P_j)} = (P_i + P_j) . \quad \textbf{(2.5)}$$

(2.5) in words: an even number , $\frac{2b}{(P_i-P_j)}$, equals the sum of two primes, $(P_i + P_j)$.

Note to find the triples P_i and P_j are defined to be products of powers of primes. Still P_i and P_j can also represent single primes: $\{P_i, P_j \in \mathbb{P}\}$.

Caveat: for $(P_i - P_j) > 0$. When $b = 0$ then $(P_i - P_j) = 0$.

Goldbach's Conjecture is one of the oldest and best known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes (www.en.m.wikipedia.org).

After inspection of **(2.5)** it appears this expression cannot be used for 4 and 6. It can be demonstrated for $\frac{2b}{(P_i-P_j)} > 6$, **(2.5)** can do the job. This equivalent to:

$$P_i \geq 5 \text{ and } P_j \geq 3 .$$

This is not a proof? To prove **(2.5)** do we need a general expression for deriving primes? This expression is still not there.
However, what we learn from **(2.5)** is: an even number can be expressed as the sum of two primes. In general:

$$2(n+1) = (P_i + P_j)\ , \qquad \textbf{(2.6)}$$

where

$\{n \in \mathbb{N} | n > 0\}$.

Goldbach's Conjecture solved?
Well, with an infinite number of primes we find an infinite number of even integers. We have all of them? That is the question.

§4 Discussion and Conclusions.

In the section on Road Map we choose:

$$c + b = 2^{2l-k} P_i^2, \qquad \textbf{(1.6)}$$

$$c - b = 2^k P_j^2. \qquad \textbf{(1.7)}$$

$$c = 2^{2l-k-1} P_i^2 + 2^{k-1} P_j^2, \qquad \textbf{(1.8)}$$

$$b = 2^{2l-k-1} P_i^2 - 2^{k-1} P_j^2\ . \qquad \textbf{(1.9)}$$

Or more concise:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^{2l-k-1} P_i^2 \\ 2^{k-1} P_j^2 \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}.$$

On the other hand we could have chosen:

$c + b = 2^r P_i^2$, and **(3.1)**

$$c - b = 2^{2l-r} P_j^2. \qquad \textbf{(3.2)}$$

Giving:

$$c = 2^{r-1} P_i^2 + 2^{2l-r-1} P_j^2, \qquad \textbf{(3.3)}$$

$$b = 2^{r-1} P_i^2 - 2^{2l-r-1} P_j^2\ , \qquad \textbf{(3.4)}$$

and

$\{r \in \mathbb{N} | 0 < r < 2l\}$.

Or more concise:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^{r-1} P_i^2 \\ 2^{2l-r-1} P_j^2 \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}.$$

In order to find triples we have the condition:

$r - 1 = 0$ or $2l - r - 1 = 0$. **(3.5)**

In addition the following constraint applies:

$b > 0$. With **(3.4)** we obtain:

$$2^{r-1} P_i^2 > 2^{2l-r-1} P_j^2 \ \rightarrow \frac{P_i}{P_j} > \frac{2^l}{2^r}\ . \qquad \textbf{(3.6)}$$

We expect the distribution of the prime numbers given in **(3.1)** and **(3.2)** to produce the same results for the Triples.

Well, comparing the conditions **(3.5)** with Eqs. **(1.8)** and **(1.9)** we see:

$(r-1)$ to be equivalent with $(2l-k-1)$,

and

$(2l-r-1)$ to be equivalent with $(k-1)$.

Plug $r-1=0$ into **(3.6)** we find

$\frac{P_i}{P_j} > \frac{2^l}{2} (= 2^{(l-1)})$. **(3.7)**

Now, plug $(2l-k-1)=0$ into

$2^{2l-k-1}P_i^2 > 2^{k-1}P_j^2 \rightarrow \frac{P_i}{P_j} > \frac{2^k}{2^l}$, **(1.11)**

and we have:

$\frac{P_i}{P_j} > \frac{2^{2l-1}}{2^l} (= 2^{l-1)})$,

equivalent to **(3.7)** .

Plug $2l-r-1=0$ into **(3.6)** we find

$\frac{P_i}{P_j} > \frac{2^l}{2^{2l-1}} (= 1/2^{(l-1)})$. **(3.8)**

Now, plug $(k-1=0)$ into

$2^{2l-k-1}P_i^2 > 2^{k-1}P_j^2 \rightarrow \frac{P_i}{P_j} > \frac{2^k}{2^l}$, **(1.11)**

then $\frac{P_i}{P_j} > \frac{1}{2^{l-1}}$

equivalent to **(3.8)**.

Hence, we find with the distribution of the prime numbers given in **(3.1)** and **(3.2)** the same Triples indeed.

For creating the Pythagorean Triples you need an efficient algorithm for producing the prime numbers: *Generating Primes* , www.en.m.wikipedia.org .

However, in general we can formulate a Prime Number Theorem:

Any odd number $k=2n-1, \{n \in \mathbb{N}, n>0\}$ is a prime number unless k can be divided by a prime number $P_j < k$, the Fundamental Theorem of Arithmetic.

The hard work begins with finding an efficient algorithm for producing the prime numbers. But that is not the point. It is about the formulation of the Theorem on Prime Numbers.

Question: how many triples can we obtain for a given value of a ?

Well, for $a=2^l$ and $\prod_s p_s^{n_s}$=1 , and for a particular value of $l \geq 2$ one triple is found. See *Table* at the end of section "A road map for generating Pythagorean triples". This is equivalent to the statement: for $n_s=0$, one triple can be found.

Now we choose $a=2^l p_t^{n_t}$, i.e. p_t one odd prime number and $n_t \geq 1$. We find at least one triple. Since for $P_i = p_t^{n_t}$ and $P_j = 1$: $\frac{P_i}{P_j} > \frac{1}{2^{l-1}}$.

Is there another triple? We have two possibilities:

$k = 1$, then $P_i = 1$ and $P_j = p_t^{n_t}$,
and
$k = 2l - 1$, then $P_i = p_t^{n_t}$ and $P_j = 1$.
We analyse $k = 1$:
Plug into $\frac{P_i}{P_j} > \frac{1}{2^{l-1}}$ the values for P_i and P_j:
$l > 1 + \frac{n_t \ln(p_t)}{\ln 2}$.
Next $k = 2l - 1$:
$l < 1 + \frac{n_t \ln(p_t)}{\ln 2}$.
We conclude to find another triple for $l > 1 + \frac{n_t \ln(p_t)}{\ln 2}$ or $l < 1 + \frac{n_t \ln(p_t)}{\ln 2}$.
Hence: for $a = 2^l p_t^{n_t}$, i.e. one odd prime number and $n_t \geq 1$, we obtain two Pythagorean triples.
Conjecture: for a given a the number of triples equals 2^s,
where s is the number of odd-coprime-triples of which a is composed.
Note: we use the wording “co-prime”, since for example 5^2 is a separate number in the triple of odd prime number series.

You will notice not to find, e.g., a combination such as: (9, 12, 15). This is a non-primitive triple. Well, set $l = 2$, $P_i = P_j$, and $a = 12$ you will find this non-primitive solution. In this case by $P_i = P_j (\neq 1)$ a non-primitive solution has been forced. A simpler approach is by multiplying the primitive Triple (3, 4, 5) by any positive integer you like. The Triple (4, 3, 5) is found with $n_s = 0$ in **(1.4)** giving P_i and P_j equal 1 in **(1.8)** and **(1.9)** .

In §3 as a spin-off of Pythagorean Triples, sort of, a solution for Goldbach’s Conjecture is found?