FERMAT's Last Theorem and PYTHAGORAS.

Proof by Contradiction.

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Leen Noordzij

Apeldoorn, The Netherlands

Dr.l.noordzij@leennoordzij.nl
www.leennoordzij.me

Content

§1. Introduction.

'Fermat's equation:

$$x^n + y^n = z^n \quad , \tag{1}$$

has no solutions for $n \geq 3'$.

A statement by Andrew Wiles written on the black board after the presentation of the conclusion concerning equation **(1).**

Is there another proof of the last theorem of Fermat, fitting into the margin of Fermat's copy of Diophantus?

It all started with $n = 2$ and

$$a^2 + b^2 = c^2. \tag{2}$$

For equation **(2)** there are an infinite number of solutions for a, b and c as positive integers; $\{a, b \text{ and } c \in \mathbb{N} | a, b \text{ and } c > 0\}$.

The question is: can we use these solutions to prove that there are no integer solutions for

$n \geq 3$? We come to that later. For $n = 3$ and $n = 4$ this proof has been given with the so called method of 'descente infinie' (Giorello, G., et al).
In §2 we proof Fermat's Last Theorem by the proof of contradiction. A so-called Fermat Triple is presumed to exist and the nonexistence is proven. The question to be answered is to find out whether this proof is sufficient. In §4 and §5 we dealt with this. Some alternative approaches for proving the nonexistence of a Fermat Triple are given in §6.

§2. Fermat's last theorem and proof by contradiction.

In general we assume the existence of a solution of **(1)** in the form of what we denominate a "Fermat" triple $x = A, y = B$ and $z = C$. A, B and C being positive integers. A is an even integer and B and C are supposed to be odd. So $\{A \in \mathbb{N} | A > 0$ and even$\}$ and $\{B, C \in \mathbb{N} | B, C > 0$ and odd$\}$; $\{n \in \mathbb{N} | n \geq 3\}$.
Does this lead to the contradiction for C not being a positive integer? A and B are relative or co-prime.
Factorise A and B , the fundamental theorem of arithmetic:
in general $A = 2^m 3^{n_1} 5^{n_2} 7^{n_3} .....p_i^{n_i} ....= 2^m \prod_{i=1}^{\infty} p_i^{n_i}$; **(3)**
$\{n_i \in \mathbb{N} \cup \emptyset\}$ and $\{p_i \in \mathbb{P}\}$,where $\mathbb{P}$ is the subset of odd prime numbers,
$\{m \in \mathbb{N} | m > 0\}$.
and $B = 3^{m_1} 5^{m_2} 7^{m_3} \dots \dots p_j^{m_j} .. = \prod_{j=1}^{\infty} p_j^{m_j}$; **(4)**
$\{m_j \in \mathbb{N} \cup \emptyset\}$ and $\{p_j \in \mathbb{P}\}$,
and $i \neq j$.
In this proof we start with values for A and B independent of each other. Consequently any value for C will be found. The proof of contradiction below will lead to the conclusion there is no "Fermat" triple.
Remark:
Even for $n = 2$, A and B independent of each other, there does not exist a Pythagorean triple. To find a Pythagorean triple you need an additional relation as explained in *Pythagorean Triples and Goldbach's Conjecture* (Noordzij).

Choosing a concise notation we write for A:
$A = 2^m P_i \equiv 2^m \prod_{i=1}^{\infty} p_i^{n_i}$ **(5)**
where $m \geq 2$ and P_i is a product of powers of prime numbers.
Note: $\prod_{i=1}^{\infty} p_i^{n_i}$, can be equal 1.
In *Pythagorean Triples* (Noordzij) $m \geq 2$ is explained.

For B we write
$B = P_j,$ **(6)**
P_j is another product of powers of prime numbers equal to the right hand side of **(4).**
C becomes with **(5)** and **(6)**
$C = (2^{mn} P_i^n + P_j^n)^{1/n}$. **(7)**

In order to find out whether or not C can be an integer we rewrite **(7)**

$$C = P_i 2^{1/2} 2^{m-1/2} (1 + \left(\frac{P_j}{P_i 2^m}\right)^n)^{1/n}. \quad \textbf{(8)}$$

C is an integer when the expression

$2^{m-1/2}(1 + \left(\frac{P_j}{P_i 2^m}\right)^n)^{1/n}$ in **(8)** equals $2^{r/2}$ or

$$2^{m-1/2}(1 + \left(\frac{P_j}{P_i 2^m}\right)^n)^{1/n} = 2^{r/2}, \quad \textbf{(9)}$$

where r in **(9)** is odd and positive; $\{r \in \mathbb{N} | r > 0 \text{ and odd }\}$.

Alas, this contradicts one of our assumptions: C to be an odd integer.

Where does $2^{r/2}$ come from? Well, take instead of $2^{m-1/2}(1 + \left(\frac{P_j}{P_i 2^m}\right)^n)^{1/n}$ a number N.

What number should N be? With **(8)** and **(9)** we have $C = P_i 2^{1/2} N$. We assume $C \in \mathbb{N}$. So $2^{1/2}N$ has to be an integer. Consequently $N = 2^{r/2}$, $\{r \in \mathbb{N} | r \text{ is odd}\}$.

So, **(9)** leads to a contradiction.

(9) can be written as(this does not create additional information):

$$2^{mn} P_i^n + P_j^n = 2^{n\frac{r+1}{2}} P_i^n, \quad \textbf{(10)}$$

with $\{\frac{r+1}{2} \in \mathbb{N} | r + 1 > 0 \text{ and even}\}$.

The left-hand side of **(10)** is odd and the right-hand side is even. Consequently the expression **(10)** is not true and leads to a contradiction and C is not an integer.

(8) can be written in a more general way :

$$C = P_i 2^{k/2} 2^{m-k/2} (1 + \left(\frac{P_j}{P_i 2^m}\right)^n)^{1/n},$$

where

$\{k \in \mathbb{N} | k > 0 \text{ and odd}\}$.

$1 \leq k \leq 2m - 1$. Clearly, for $m = 1, k = 1$.

Furthermore $r + k > 2m$.

Note: For $r + k = 2m$, we find a trivial solution: $B = 0$ and $A = C$.

(9) becomes:

$$2^{m-k/2}(1 + \left(\frac{P_j}{P_i 2^m}\right)^n)^{1/n} = 2^{r/2}$$

and **(10)** :

$$2^{mn} P_i^n + P_j^n = 2^{n\frac{r+k}{2}} P_i^n.$$

Again, for the latter expression: the left-hand side is odd and the right-hand side is even. No surprise here: C is even.

This proof of contradiction is necessary. Is it sufficient? This is more or less a philosophical question. Is it? A proof by contradiction is necessary and sufficient.

However, set $n = 2$ in **(10)** :

$$2^{m2} P_i^2 + P_j^2 = 2^{2\frac{r+1}{2}} P_i^2,$$

the left hand-side is odd and the right-hand side is even. A contradiction: no Pythagorean triples. So, in general you will not find triples. We need something in addition. For convenience I summarize the results of *Pythagorean Triples* (Noordzij).

(2) can be written as

$(c-b)(c+b)=a^2$.

In the block below I reproduced the results of *Pythagorean Triples* (Noordzij):

We write a as a product of prime numbers, p_s:

$a \coloneqq 2^l 3^{n_1} 5^{n_2} 7^{n_3} \dots \dots p_s^{n_s} \dots \dots$, **(1.2)**

with $n_1, n_2, \dots \dots, n_s \dots$, ≥ 0 .

So $a = 2^l \prod_{s=1}^{\infty} p_s^{n_s}$.

With $\{n_s \in \mathbb{N} \cup 0\}$ and $\{l \in \mathbb{N} | l > 0\}$.

p_s belongs to the subset of odd prime numbers $\mathbb{P}$: $\{p_s \in \mathbb{P} |\ p_s \text{ odd}\}$ and $\mathbb{P} \subset \mathbb{N}$. Furthermore: $p_{s+1} > p_s$.

b en c can be expressed in a, with (**1.1**) , :

$(c+b)(c-b)=a^2$. **(1.3)**

$c+b$ and $c-b$, $\{c \pm b \in \mathbb{N} | c \pm b > 0 \text{ and even}\}$.

Euclid proofed the existence of an infinite number of Pythagorean triples.

$c+b$ and $c-b$ are integer factors of a^2. For example: with $a=4$, $c+b=8$ and $c-b=2$.

With Eq. (**1.3**) we have $c+b<a^2$, since $c-b>1$.

Also $c-b<a^2$, since $c+b>1$.

With $c+b$ and $c-b$ to be even positive integers, $c+b$ and $c-b$ are integer factors of a^2 indeed. So $c+b=\frac{a^2}{c-b}$, and $c-b=\frac{a^2}{c+b}$.

For convenience we write $a = 2^l \prod_{s=1}^{\infty} p_s^{n_s} = 2^l P_i P_j$, **(1.4)**

where P_i and P_j are products of powers of odd prime numbers. P_i and P_j being co-prime:

$P_i \cap P_j = \emptyset$ and $\{P_i , P_j \subseteq \prod_{s=1}^{\infty} p_s^{n_s}\}$.

For $n_s = 0$: $P_i = 1$ and $P_j = 1$.

For $n_s \neq 0$: a possible combination of $\{P_i , P_j\}$ is:

$P_i = \prod_{s=1}^{\infty} p_s^{n_s}$ and $P_j = 1$,

and

$P_j = \prod_{s=1}^{\infty} p_s^{n_s}$ and $P_1 = 1$.

Furthermore: $P_i < P_j$ or $P_j < P_i$ **.**

The lowest possible triple is obtained for $n_s = 0$ and $l = 2$: (4, 3, 5).

For any combination of b and c, P_i en P_j can be found from different combinations of products of powers of prime numbers given in Eq. (**1.2**).

In general, with **(1.4)**, we have:

$(c+b)(c-b)=a^2=2^{2l}P_i^2P_j^2$. **(1.5)**

Since b and c are co-prime note the above representation of a in **(1.4)** is not just for convenience. With **(1.5)** , $c+b$ and $c-b$ can be represented respectively:

$c+b=2^{2l-k}P_i^2$, **(1.6)**

$c-b=2^kP_j^2$. **(1.7)**

For c and b we have:

$c=2^{2l-k-1}P_i^2+2^{k-1}P_j^2$, **(1.8)**

$b=2^{2l-k-1}P_i^2-2^{k-1}P_j^2$. **(1.9)**

$\{k \in \mathbb{N} | 0 < k < 2l\}$.

Keep in mind: $c>b$ and c, b odd. Consequently $c-b>1$ and $(c+b)<2^{2l}P_i^2P_j^2$.

To make c and b co-prime indeed, we need to do something additional, unless:

$k-1=0$ or $2l-k-1=0$, **(1.10)**

we do not obtain a primitive Triple.

With $k=1$, $2l-k-1$ has to be larger than 0, or $2l-1-1>0 \rightarrow l>1$,

and with

$2l-k-1=0 \rightarrow k=2l-1 \rightarrow k-1=2l-2$,

hence $l>1$.

A brief summary of the above block can be found in paragraph 4.

In the following paragraphs further attention will be paid to the proof of the (non)existence of Fermat triples. In the following paragraph prime numbers are discussed.

§3. A Goldbach Interlude.

Does **(10)** and $2^{mn}P_i^n + P_j^n = 2^{n\frac{r+1}{2}}P_i^n$ ring a bell? Not at first sight since the equality sign is misleading. The sign is wrong.
We can set $m = 0,\ \ n = 1$, we substitute P_k for P_i in the right-had side of **(10)** and set $\frac{r+1}{2} = l, \{\, l \in \mathbb{N}\}$, this results into a new expression:

$$P_i + P_j = 2^l P_k. \qquad \textbf{(10a)}$$

P_k can be a single prime or a product of prime numbers.
For example we can set $P_i = P_j = P_k = 2$ and $r = 1(l = 1)$, we find $2 + 2 = 4$. Well, this looks almost trivial.
Now we choose l= 2, $P_k = 2$ and we are free to choose P_i and P_j to be 3 and 5 respectively. Then $3 + 5 = 8$. In the following analysis I set P_i and P_j to be odd prime numbers. This is consistent with the notation in the other paragraphs.
Are we exploring Goldbach's Conjecture (Noordzij)?
With $l = 2$, $P_k = 3$ and with the free choice of P_i and P_j to be 5 and 7 respectively. Then 5+7 $= 12$.
Now choose P_i and P_j to be 23 and 37 respectively. Then with **(10a)** , $2^l P_k = 60$. Plug $l = 2$ into this expression and we have $P_k = 15$, a product of two prime numbers.
Actually **(10a)** can be written as $P_i + P_j = 2n$, $\{n \in \mathbb{N} | n > 2\}$ and $\{P_i, P_j \in \mathbb{P}\}$.
Furthermore we have $2 < P_i \leq n$ and $n \leq\ P_j < 2n - 2$.
In addition, with **(10a)**, $P_j = 2n - P_i$. So, we have $2 < P_i \leq n$ or $0 \leq n - P_i < n - 2$. This indicates you can always find the prime number P_i and consequently the prime number P_j for any n with $\{n \in \mathbb{N} | n > 2\}$.
Proof by contradiction: Suppose you cannot find a prime number P_i for $2 < P_i < n$ and $\{n \in \mathbb{N} | n > 2\}$. This contradicts the existence of prime numbers. Suppose you cannot find a prime number P_j for $n \leq\ P_j < 2n - 2$. This contradicts again the existence of prime numbers.
An almost trivial solution for Goldbach's conjecture is n to be a prime number. Then a solution $P_i = P_j = n$ can be found. A complete picture for $2n = 34$ is given in the table below:

$2n = 34$	$n = 17$
P_i	P_j
3	31
5	29
11	23
17	17

Keep in mind: **(10a)** cannot be applied in §2 to find out about the contradiction.
We just explored Goldbach's Conjecture(See Noordzij).

§4. Caveat Triplus.

As a test we have set $n = 2$ in paragraph 2 and find the contradiction again as it should be. Since in general with $n = 2$ in **(1)** we will find no Pythagorean triple: A and B are relatively prime or coprime and independent. In order to find a Pythagorean triple (A, B and C), B and C can no longer to be considered independent (Noordzij, we used a, b and c as a Pythagorean triple). There we derived the relations for Pythagorean triples. In the notation of that article the additional relation to be used is: $(c + b)(c - b) = 2^{2m}P_i^2P_j^2$.
P_i and P_j are combinations of odd prime numbers with $P_i \neq P_j$. They are coprime or relative prime.
This leads to $c = (2P_i^2 + 2^{2m-1}P_j^2)/2$ and $b = (2P_i^2 - 2^{2m-1}P_j^2)/2$. You will find for example the following Triple: (21,220,221). In this example $a = 220$.

§5. Observations about the proof of contradiction for $n > 2$.

In paragraph 2 we found the proof by contradiction to be not sufficient. Do we need in addition a relation between B and C?
Can we still confirm the proof by contradiction for the "Fermat" triple A, B and C, for $n \geq 3$?
Let us look into it in more detail.
With A, B and C assumed to be a triple $\{A, B$ and $C \in \mathbb{N}\}$, a "Fermat" triple, **(1)** is written as:
$A^n + B^n = C^n$ or

$$C^n - B^n = A^n. \qquad \textbf{(11)}$$

Then **(11)** gives:

$$(C - B)(\textstyle\sum_{k=0}^{n-1} C^{n-1-k} B^k) = A^n . \qquad \textbf{(12)}$$

Proof of $(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = C^n - B^n$:
$(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = \sum_{k=0}^{n-1} C^{n-k} B^k - \sum_{k=0}^{n-1} C^{n-1-k} B^{k+1}$.
$\sum_{k=0}^{n-1} C^{n-k} B^k - \sum_{k=0}^{n-1} C^{n-1-k} B^{k+1} = C^n + \sum_{k=1}^{n-1} C^{n-k} B^k - B^n - \sum_{k=0}^{n-2} C^{n-1-k} B^{k+1}$.
Now change the summation index of $\sum_{k=1}^{n-1} C^{n-k} B^k$, $k - 1 = r$ and we have:
$C^n - B^n + \sum_{r=0}^{n-2} C^{n-1-r} B^{r+1} - \sum_{k=0}^{n-2} C^{n-1-k} B^{k+1}$.
Hence $\sum_{r=0}^{n-2} C^{n-1-r} B^{r+1} - \sum_{k=0}^{n-2} C^{n-1-k} B^{k+1} = 0$.
Consequently $(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = C^n - B^n$.
End of Proof.

Now with the presumption of A, B and C being a "Fermat" triple, $(C - B)$ and the summation between brackets in **(12)** are positive integers. Both $(C - B)$ and the summation in **(12)** can be expressed in fractions of A^n, consisting of products of powers of prime numbers.

Proof:

Let $(C-B)$ be no integer factor of A^n. Consequently $\frac{A^n}{(C-B)}$ is not an integer. So C and/or B in **(12)** cannot be an integer. For a Fermat triple $(C-B)$ has to be a factor of A^n.

End of proof.

$(C-B)$ is a positive integer, even and a factor of A^n, A_1 say $\{A_1 \in \mathbb{N} | A_1 > 0$ and even$\}$.

The summation in **(12)** constitutes the other factor of A^n, A_3 say.

In addition $(C+B)$ is also a positive integer, even and smaller than A_3, A_2 say $\{A_2 \in \mathbb{N} | 0 < A_2 < A_3, A_2$ is even$\}$.

Furthermore $A_1 < A$.

Proof:

$C - B = A_1$,

and $C^n = (B + A_1)^n = \sum_{k=0}^{n} \binom{n}{k} {A_1}^k B^{n-k} = B^n + A_1^n + \sum_{k=1}^{n-1} \binom{n}{k} {A_1}^k B^{n-k} = B^n + A^n$.

$A_1^n + \sum_{k=1}^{n-1} \binom{n}{k} {A_1}^k B^{n-k} = A^n$.

So $A_1^n < A^n$ and $A_1 < A$.

End of Proof.

Substitution of A_1 into **(11)** a polynomial in B can be found.

$(B + A_1)^n - B^n = A^n$,

and

$\sum_{k=0}^{n} \binom{n}{k} {A_1}^k B^{n-k} = A^n + B^n$.

The polynomial in B is:

$$B^{n-1} + \frac{1}{n} \sum_{k=2}^{n-1} \binom{n}{k} {A_1}^{k-1} B^{n-k} - \frac{1}{n}\left(\frac{A^n}{A_1} - A_1^{n-1}\right) = 0. \quad \textbf{(13)}$$

The polynomial has positive roots for $\frac{1}{n}\left(\frac{A^n}{A_1} - A_1^{n-1}\right) > 0$.

Since $A_1 < A$ the above inequality is true. Is the left hand-side of the inequality an integer? If so, one of the roots B is a positive integer (Wijdenes).

§6. Proving the (non)existence of a Fermat Triple by proof of contradiction.

First approach.

As mentioned before, the proof by contradiction in §2, is not sufficient. Let's take another look at Fermat's theorem.

Presume again a Fermat triple to exist.

$C - B (\equiv A_1)$ in **(12)** a positive integer, even and a factor of A^n, $\{C - B \in \mathbb{N} | C - B > 1$ and even$\}$.

The summation in **(12)** $(\sum_{k=0}^{n-1} C^{n-1-k} B^k \equiv A_3)$ constitutes the other integer factor of A^n since A_1 constitutes a factor of A^n. For n odd, A_3 is odd and for n even A_3 is even.

In addition $(C+B\equiv A_2)$ $\{A_2\in\mathbb{N}|A_2<A_3$, and A_2 is even$\}$.   (Except for $n=2$: $A_2=A_3$).
So $C+B\leq\sum_{k=0}^{n-1}C^{n-k-1}B^k$.
Since we are free to choose any positive even integer for A we choose $(C+B)$ to be a factor of A^n.
Does this create a constraint?
For n is even $(C+B)$ is an integer factor of $A_3\equiv\sum_{k=0}^{n-1}C^{n-1-k}\,B^k$:
$\sum_{k=0}^{n-1}C^{n-1-k}\,B^k=(C+B)\sum_{k=0}^{(n-2)/2}C^{n-2-2k}\,B^{2k}$.
We pay attention to that later on. So $(C+B)$ is an integer factor of A^n.

Keep in mind: A^n contains at least a factor 2^{2n}, in order to find in the limit $n=2$ Pythagorean Triples (Noordzij).

With $C+B=A_2$ and $C-B=A_1$
$C=(A_2+A_1)/2$,
$B=(A_2-A_1)/2$.
C and B are odd. We choose A_2 to contain a factor 2. Consequently A_1 contains at least 2^2. Besides A_1 and A_2 do not have other common factors, lest we obtain trivial, non-primitive solutions. Does this lead to a contradiction?
Since $C+B$ is even and a factor of A^n, $\frac{(C+B)}{2}$ (odd) has to be a factor of A_3 for n is odd.
Note: A_3 is a factor of $A^n$and A_2 contains a factor 2.
After inspection you will find $\frac{(C+B)}{2}$ not to be a factor of $A_3(\equiv\sum_{k=0}^{n-1}C^{n-1-k}\,B^k)$ for n is odd.
There remains a residue equal to:
$\frac{2B^{n-1}}{C+B}$.
This is a contradiction and there is no Fermat triple for n is odd.
Now let us consider whether A_2 to be a factor of A^n.
Substitute $C=(A_2+A_1)/2$ and $B=(A_2-A_1)/2$ into **(11)**

$$\sum_{k=0}^{n}\binom{n}{k}{A_1}^k{A_2}^{n-k}-\sum_{k=0}^{n}\binom{n}{k}(-A_1)^k{A_2}^{n-k}=2^nA^n. \quad \textbf{(14)}$$

For k even the left hand side of **(14)** is 0.
For k is odd, **(14)** results into

$$\sum_{k=1}^{n}\binom{n}{k}{A_1}^k{A_2}^{n-k}=\ 2^{n-1}A^n. \quad \textbf{(15)}$$

Here we have a polynomial for A_2:

$$A_2^n+\sum_{k=1}^{n-2}\binom{n}{k}{A_1}^k{A_2}^{n-k}-(2^{n-1}A^n-A_1^n)=0. \quad \textbf{(16)}$$

(16) gives a positive integer root for A_2 when A_2 is a integer factor of $(2^{n-1}A^n-A_1^n>0)$. (Wijdenes).
n is odd, A_1 contains all the factors 2 of A^n. See equation **(12)**. As been mentioned before A_2 contains a factor 2 and no common factors with A_1. Through formally expanding A, A_1 and A_2 in powers of prime numbers, P, Q and R (short-hand notation and relative prime or

coprime) we have:

$$\left.\begin{array}{l} A = 2^l PR \text{ and } A^n = 2^{ln} P^n R^n, \\ A_1 = 2^{ln-1} P_1 \text{ and} \\ A_2 = 2Q. \end{array}\right) \quad \textbf{(17)}$$

Where P_1 is a integer factor of P^n.
$\{l \geq 2$ and $l \in \mathbb{N}\}$.
A_2 is an integer factor of $(2^{n-1}A^n - A_1^n)$. Then $(2^{n-1}A^n - A_1^n)/A_2$ has to be an integer . By substituting A, A_1 and A_2 of **(17)** into $(2^{n-1}A^n - A_1^n)/A_2$ there remains a quotient of an even and an odd factor. So there is no positive integer root for A_2 in **(16)** and by proof of contradiction there is no Fermat triple.
We have investigated n is odd.

For n is even :
$n = 2(s+1)$ with $s = 1(1)$ … and we preclude $n = 2^s$ with $s = 2(1)$ …
(12) can be written:

$$(C^{\frac{n}{r}} + B^{\frac{n}{r}}) \dots \dots \dots \left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)(C - B)\left(\sum_{p=0}^{\frac{n}{q}-1} C^{\frac{n}{q}-1-p} B^p\right) = A^n. \quad \textbf{(18)}$$

With $r = 2(2) \dots . q$. We stop factorizing $(C^n - B^n)$ for the first value of $^n/_q$ odd and an integer. $\left(C^{\frac{n}{r}} + B^{\frac{n}{r}}\right)$, ….. … …, and $\left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)$ are all even and factors of A^n. The remaining factor, A_4 say, equals $(C - B)\left(\sum_{p=0}^{\frac{n}{q}-1} C^{\frac{n}{q}-1-p} B^p\right)$.
We apply the above approach for ***n* is odd** to

$$(C - B)\left(\sum_{p=0}^{\frac{n}{q}-1} C^{\frac{n}{q}-1-p} B^p\right) = C^{\frac{n}{q}} - B^{\frac{n}{q}}. \quad \textbf{(19)}$$

$$C^{\frac{n}{q}} - B^{\frac{n}{q}} \equiv A_4, \quad \textbf{(20)}$$

is a factor of A^n.
A_4 is even, and since $^n/_q$ is odd, $C - B(= A_1)$ contains all the factors 2 of A_4.
A_2 contains a factor 2 and no other common factors with A_1. Otherwise we find trivial non-primitive solutions.
Substitute $C = (A_2 + A_1)/2$ and $B = (A_2 - A_1)/2$ into **(20)** we have

$$A_2^{n/q} + \sum_{k=1}^{n/q-2} \binom{n/q}{k} A_1{}^k A_2{}^{n/q-k} - \left(2^{n/q-1}A_4 - A_1^{n/q}\right) = 0. \quad \textbf{(21)}$$

Again, expanding A, A_1, A_2 and A_4 in to powers of prime numbers we find no Fermat triple.

In **§1** the method of ‘descente infinie’ is mentioned**.**
With the approach of expanding $C^n - B^n = A^n$ (**§2**) we can find another proof of the nonexistence of a Fermat triple for $n = 3$ and $n = 4$.
$n = 3$:
We substitute $C = B + A_1$ into **(11)**. The result is a quadratic equation in B. There is one

possible positive integer root:

$$B = -\frac{A_1}{2} + \frac{\sqrt{3}}{6}\sqrt{(\frac{4A^3}{A_1} - A_1^2)}, \quad \textbf{(22)}$$

note: $\frac{\sqrt{3}}{6}\sqrt{\frac{4A^3}{A_1} - A_1^2} > \frac{A_1}{2}$.

When the square root in **(22)** is proportional to $3^{(2k+1)/2}$, with $k = 0(1)..$, B will be a positive integer. This condition gives a cubic equation in A_1:

$A_1^3 + 3^{2k+1}A_1 - 4A^3 = 0.$

After inspection there are: one real root and a pair of complex conjugate roots for A_1 (Abramowitz, et al). The real root is irrational: there is no Fermat triple for $n = 3$.

$n = 4$:

We find with **(11)** or **(12)**

$$(C - B)(C + B)(C^2 + B^2) = A^4. \quad \textbf{(23)}$$

After substitution of $(C - B) = A_1$ and $(C + B) = A_2$ **(23)** becomes

$$A_2A_1(A_2^2 + A_1^2) = 2A^4. \quad \textbf{(24)}$$

For a given value of A and a given value of A_1 , there are: one real root and a pair of complex conjugate roots for A_2 of the cubic equation

$A_2^3 + A_2A_1^2 = \frac{A^4}{A_1}$. (Abramowitz, et al).

The real root is irrational and there is no Fermat triple for $n = 2^2$.

So: there is no Fermat triple for $n = 4$ proved by means of 'descente infinie' and here through the above complex value for A_2.

What about $n > 4$? **(11)** can be written as:

$$(A^{n/4})^4 + (B^{n/4})^4 = (C^{n/4})^4. \quad \textbf{(25)}$$

With the result for $n = 4$ and **(25)** there is no Fermat triple for $n = 4s$ with s= $1(1)$ ….. This includes $n = 2^s$ for s≥ 2.

Note: for $n = 2^r$, $r = 2(1)$ … **(12)** can be written as

$(C - B)\prod_{k=0}^{r-1}(C^{2^k} + B^{2^k}) = A^n$.

The second approach.

Let us assume the existence of a triple a, b and c. $\{a, b$ and $c \in \mathbb{N}\}$. Starting with a and b independent, in **(§1)** it has been shown there are no integer solutions for c with $n \geq 2$ by proof of contradiction. That proof is not sufficient.

When $\{b$ and $c \in \mathbb{N}\}$ and $n = 2$ both b and c can be explicitly expressed into factors of a^2 (Noordzij).

For $n \geq 3$, $c - b$ is formulated as a factor of a^n . Consequently, a polynomial in b or c results as the other factor of a^n.

Now we will use the non-trivial or primitive solutions of **(2)** in order to prove **(1)** has no integer solutions. We rewrite **(1)** as

$$(x^{(\frac{n}{2})})^2 + (y^{(\frac{n}{2})})^2 = (z^{(\frac{n}{2})})^2. \quad \textbf{(26)}$$

We have **(1)** formulated as a Pythagorean equation.
For a Pythagorean triple, a, b and c from **(2),** it is known that two of the triple are uneven and one is even. In general we may choose $a = x^{(\frac{n}{2})}$ to be such an even positive integer that this will lead to non-trivial or primitive solutions of **(2)** . We also choose x to be an integer otherwise there is nothing left to be proven. b and c are odd.
We factorize $a = 2^l P_i P_j$ (Noordzij). Then:

$$a^2 = 2^{2l} P_i^2 P_j^2 \, , \tag{27}$$

where $l \geq 2$ giving non-trivial solutions of **(2)** .
For x to be an integer $2l$ has to be a multiple of n and P_i en P_j have to be products of powers n of prime numbers ≥ 3. P_i and P_j are mutual prime.
With (Noordzij) b and c are :

$$b = 2^{2l-2} P_i^2 - P_j^2 \text{ and} \tag{28}$$
$$c = 2^{2l-2} P_i^2 + P_j^2. \tag{29}$$

For y and z we find using **(26)**, **(28)** and **(29)**:

$$y = (2^{2l-2} P_i^2 - P_j^2)^{2/n}, \tag{30}$$

$$z = (2^{2l-2} P_i^2 + P_j^2)^{2/n}. \tag{31}$$

We cannot conclude, using **(30)** and **(31)**, y or z to be irrational. In order to find out about the existence of a Fermat triple we write for y

$$y = (2^{2l-2} P_i^2)^{2/n} (1 - \frac{P_j^2}{P_i^2 2^{2l-2}})^{2/n}. \tag{32}$$

In **(32)** $(P_i)^{2/n}$ is an integer.
What about $(2^{2l-2})^{2/n} (1 - \frac{P_j^2}{P_i^2 2^{2l-2}})^{2/n}$?
For y to be a positive integer $(1 - \frac{P_j^2}{P_i^2 2^{2l-2}})^{2/n}$ has to be equal to $(2^{2l-2})^{-2/n}$.
Then $P_i^2 (2^{2l-2} - 1) = P_j^2$.
We conclude P_i^2 and P_j^2 not to be mutual prime.
In **(32)** we see also y to depend on $(P_i)^{\mathbf{2/n}}$, so only trivial solutions exist for $n \geq 3$. There is no Fermat triple.
Note: P_i and P_j not mutually prime contradicts the assumption.

The above method can be applied for the other combinations of non-trivial solutions of **(2)** as mentioned in (Noordzij).

For further reading on Fermat's last theorem I like to mention Simon Singh's book.

Finally to conclude the above approach on Fermat's last theorem we like to cite Feynman on Fermat's last theorem: "For my money Fermat's theorem is true". Feynman estimated that the probability of finding integer solutions is less than 10^{-200} (Schweber).

§7. Conclusions.

The nonexistence of a Fermat triple has been proven by the proof of contradiction in §5 and 6. In §5 and 6 we showed the need of additional conditions. On basis of these results we finally may conclude: there exist no Fermat Triples.

§8. Literature:

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