Pythagorean Triples

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Content

A Roadmap for generating Pythagorean Triples.

To find the solutions, represented by prime numbers, of

$$a^2 + b^2 = c^2, \qquad \textbf{(1.1)}$$

where a, b en c are positive integer numbers, $\{a, b, c \in \mathbb{N} | a, b, c > 0\}$, is a standard procedure.
Instructive reading on Pythagorean Triples can be found, e.g., in www.en.wikipedia.org .
a, b en c constitute a so-called Pythagorean Triple. In general a is even and b and c are odd. The approach we choose is based on The Fundamental Theorem of Arithmetic.
The Theorem: any integer greater than 1 is either a prime number, or can be written as a unique product of prime numbers.
We write a as a product of prime numbers, p_s:

$$a \coloneqq 2^l 3^{n_1} 5^{n_2} 7^{n_3} \dots \dots p_s^{n_s} \dots \dots\dots, \qquad \textbf{(1.2)}$$

with $n_1, n_2, \dots \dots, n_s \dots.\ , \geq 0$.
So $a = 2^l \prod_{s=1}^{\infty} p_s^{n_s}$.
With $\{n_s \in \mathbb{N} \cup 0\}$ and $\{l \in \mathbb{N} | l > 0\}$.
p_s belongs to the subset of odd prime numbers $\mathbb{P}$: $\{p_s \in \mathbb{P} |\ p_s$ odd$\}$ and $\mathbb{P} \subset \mathbb{N}$.
Furthermore: $p_{s+1} > p_s$.
b en c can be expressed in a, with (**1.1**) , :

$$(c + b)(c - b) = a^2. \qquad \textbf{(1.3)}$$

$c + b$ and $c - b$, $\{c \pm b \in \mathbb{N} | c \pm b > 0$ and even$\}$.
Euclid proofed the existence of an infinite number of Pythagorean triples.

$c + b$ and $c - b$ are integer factors of a^2. For example: with $a = 4$, $c + b = 8$ and $c - b = 2$.
With Eq. (**1.3**) we have $c + b < a^2$, since c$-b > 1$.
Also $c - b < a^2$, since $c + b > 1$.
With $c + b$ and $c - b$ to be even positive integers, $c + b$ and $c - b$ are integer factors of a^2 indeed. So $c + b = \frac{a^2}{c-b}$, and $c - b = \frac{a^2}{c+b}$.
For convenience we write $a = 2^l \prod_{s=1}^{\infty} p_s^{n_s} = 2^l P_i P_j$, **(1.4)**
where P_i and P_j are products of powers of odd prime numbers. P_i and P_j being co-prime: $P_i \cap P_j = \emptyset$ and $\{P_i , P_j \subseteq \prod_{s=1}^{\infty} p_s^{n_s}\}$.
For $n_s = 0 : P_i = 1$ and $P_j = 1$.
For $n_s \neq 0$: a possible combination of $\{P_i , P_j\}$ is:
$P_i = \prod_{s=1}^{\infty} p_s^{n_s}$ and $P_j = 1$,
and
$P_j = \prod_{s=1}^{\infty} p_s^{n_s}$ and $P_1 = 1$.
Furthermore: $P_i < P_j$ or $P_j < P_i$ **.**

The lowest possible triple is obtained for $n_s = 0$ and $l = 2$: (4, 3, 5).
For any combination of b and c, P_i en P_j can be found from different combinations of products of powers of prime numbers given in Eq. (**1.2**).
In general, with **(1.4)**, we have:
$(c + b)(c - b) = a^2 = 2^{2l} P_i^2 P_j^2$. **(1.5)**
Since b and c are co-prime note the above representation of a in **(1.4)** is not just for convenience. With **(1.5)** , $c + b$ and $c - b$ can be represented respectively:
$c + b = 2^{2l-k} P_i^2$, **(1.6)**
$c - b = 2^k P_j^2$. **(1.7)**
For c and b we have:
$c = 2^{2l-k-1} P_i^2 + 2^{k-1} P_j^2$, **(1.8)**
$b = 2^{2l-k-1} P_i^2 - 2^{k-1} P_j^2$. **(1.9)**
$\{k \in \mathbb{N} | 0 < k < 2l\}$.
Keep in mind: $c > b$ and c, b odd. Consequently $c - b > 1$ and $(c + b) < 2^{2l} P_i^2 P_j^2$.
To make c and b co-prime indeed, we need to do something additional, unless:
$k - 1 = 0$ or $2l - k - 1 = 0$, **(1.10)**
we do not obtain a primitive Triple.
With $k = 1$, $2l - k - 1$ has to be larger than 0, or $2l - 1 - 1 > 0 \rightarrow l > 1$,
and with
$2l - k - 1 = 0 \rightarrow k = 2l - 1 \rightarrow k - 1 = 2l - 2$,
hence $l > 1$.
So, these two constraints lead to the following condition: $l > 1$. Since for $l = 1$, and the aforementioned constraints, the right-hand sides of Eqs. **(1.8)** and **(1.9)** represent the sum of

two odd integers and become consequently even.
a , in the Triples you are looking for, is at least an integer that can be divided by 4.
Again: for $l = 2, k = 1$ we find the Triple (4, 3,5).
We also need the constraint: $b > 0$. With **(1.9)** we obtain:

$$2^{2l-k-1}P_i^2 > 2^{k-1}P_j^2 \rightarrow \frac{P_i}{P_j} > \frac{2^k}{2^l} \ . \qquad \textbf{(1.11)}$$

To summarize:

$$a = 2^l \prod_{s=1}^{\infty} p_s^{n_s} \equiv 2^l P_i P_j \ ,$$

and
Table Roadmap:

k	P_i	P_j	$\frac{P_i}{P_j}$	$\frac{2^k}{2^l}$	Triple	$+\rightarrow c$ $-\rightarrow b$
1	$\prod_s p_s^{n_s}$	1	$\prod_s p_s^{n_s}$	$\frac{1}{2^{l-1}}$	yes	$2^{2l-2}P_i^2 \pm 1$
1	$\prod_i p_i^{n_i}$ $i \neq j$	$\prod_j p_j^{n_j}$ $j \neq i$	$\frac{\prod_i p_i^{n_i}}{\prod_j p_j^{n_j}}$	$\frac{1}{2^{l-1}}$	?	$2^{2l-2}P_i^2 \pm P_j^2$
1	1	$\prod_s p_s^{n_s}$	$\frac{1}{\prod_s p_s^{n_s}}$	$\frac{1}{2^{l-1}}$	?	$2^{2l-2} \pm P_j^2$
1	1	1	1	$\frac{1}{2^{l-1}}$	yes	$2^{2l-2} \pm 1$
$2l-1$	$\prod_s p_s^{n_s}$	1	$\prod_s p_s^{n_s}$	2^{l-1}	?	$P_i^2 \pm 2^{2l-2}$
$2l-1$	$\prod_i p_i^{n_i}$ $i \neq j$	$\prod_j p_j^{n_j}$ $j \neq i$	$\frac{\prod_i p_i^{n_i}}{\prod_j p_j^{n_j}}$	2^{l-1}	?	$P_i^2 \pm 2^{2l-2}P_j^2$
$2l-1$	1	$\prod_s p_s^{n_s}$	$\frac{1}{\prod_s p_s^{n_s}}$	2^{l-1}	no	
$2l-1$	1	1	1	2^{l-1}	no	

Note: the minimum value of $l = 2$.

Pythagorean Triples.

Let us start with $l = 2, n_s = 0$ and with **(1.10)** $k = 1$ or $k = 3$.
- $k = 1$.
We have $a = 4$,

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 1 > \frac{2^k}{2^l} = \frac{1}{2}$.

With **(1.8)** and **(1.9):**

$c = 4 + 1 = 5$,

$b = 4 - 1 = 3$.

We obtain the triple we are familiar with: (4, 3, 5) .

- $k = 3$.

There is no triple. The triple is not allowed due the constraint **(1.11) .**

Now for the next triple(s): $l = 2, n_s = 1, s = 1$, $p_s = 3$ and with **(1.10)** $k = 1$ or $k = 3$.

- $k = 1$.

We have the smallest prime: 3.

$P_i = 3$ and $P_j = 1$.

Then $a = 4 \times 3 = 12$,

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 3 > \frac{2^k}{2^l} = \frac{1}{2}$.

from **(1.8)** and **(1.9)** we find:

$c = 4 \times 9 + 1 = 37$,

$b = 4 \times 9 - 1 = 35$.

The triple is: (12, 35, 37).

- $k = 3$.

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 3 > \frac{2^k}{2^l} = 2$.

So we expect another triple.

With **(1.8)** and **(1.9):**

$c = 9 + 4 = 13$,

$b = 9 - 4 = 5$.

The triple is (12, 5, 13).

So we have two triples for $a = 4 \times 3 = 12$.

However in the algorithm you need to find out about $P_i = 1$ and $P_j = 3$.

The constraint **(1.11)** forbids another triple.

Let us try another one.

Now for the next triple(s): $l = 2, n_s = 1, s = 1$, $p_s = 5$ and with **(1.10)** $k = 1$ or $k = 3$.

- $k = 1$.

We use the second smallest prime: 5 .

$P_i = 5$ and $P_j = 1$.

Then $a = 4 \times 5 = 20$,

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 5 > \frac{2^k}{2^l} = \frac{1}{2}$.

from **(1.8)** and **(1.9)** we find:

$c = 4 \times 25 + 1 = 101$,

$b = 4 \times 25 - 1 = 99.$

- $k = 3.$

with **(1.4)** and **(1.11)** $\frac{P_i}{P_j} = 5 > \frac{2^k}{2^l} = 2$.

So we expect another triple.

With **(1.8)** and **(1.9):**

$c = 25 + 4 = 29$,

$b = 25 - 4 = 21.$

The triple is (20, 21, 29).

We have two triples for $a = 20$.

For a particular value of a there are more than 2 non-trivial or primitive Pythagorean Triples.

An example: $a = 60(= 2^2.3.5)$ produces four Pythagorean Triples : (60. 11, 61) , $(60, 91, 109)$, $(60, 221, 229)$ and $(60, 899, 901)$.

So, with $a = 2^l \prod_{s=1}^{2} p_s^{n_s}$, $n_s = 1, s = 2,\ p_1 = 3$, $p_2 = 5$ and $l = 2$:

k	P_i	P_j	P_i/P_j	$2^k/2^l$	triple
1	3×5	1	15	1/2	Yes: 60, 899, 901
1	3	5	3/5	1/2	Yes: 60, 11, 61
1	5	3	5/3	1/2	Yes: 60, 109, 91
1	1	3×5	1/15	1/2	no
3	3×5	1	15	2	Yes: 60, 221, 229
3	3	5	3/5	2	no
3	5	3	5/3	2	no
3	1	3×5	1/15	2	no
					Four triples

Note:

- given $a = 180(= 2^2.3^2.5)$, so $n_1 = 2$ you will find again four triples.
- given $a = 2^2.3.5.7$, you will find eight triples.
- given $a = 2^2.3.5.7.11$, you will find sixteen triples.

Discussion and Conclusions.

In the section on Road Map we choose:

$$c + b = 2^{2l-k} P_i^2, \quad \textbf{(1.6)}$$

$$c - b = 2^k P_j^2. \quad \textbf{(1.7)}$$

$$c = 2^{2l-k-1} P_i^2 + 2^{k-1} P_j^2, \quad \textbf{(1.8)}$$

$$b = 2^{2l-k-1} P_i^2 - 2^{k-1} P_j^2 . \quad \textbf{(1.9)}$$

Or more concise:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^{2l-k-1} P_i^2 \\ 2^{k-1} P_j^2 \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix} .$$

On the other hand we could have chosen:

$c + b = 2^r P_i^2$, and **(3.1)**

$c - b = 2^{2l-r} P_j^2$. **(3.2)**

Giving:

$c = 2^{r-1} P_i^2 + 2^{2l-r-1} P_j^2$, **(3.3)**

$b = 2^{r-1} P_i^2 - 2^{2l-r-1} P_j^2$, **(3.4)**

and

$\{r \in \mathbb{N} | 0 < r < 2l\}$.

Or more concise:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^{r-1} P_i^2 \\ 2^{2l-r-1} P_j^2 \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}.$$

In order to find triples we have the condition:

$r - 1 = 0$ or $2l - r - 1 = 0$. **(3.5)**

In addition the following constraint applies:

$b > 0$. With **(3.4)** we obtain:

$2^{r-1} P_i^2 > 2^{2l-r-1} P_j^2 \rightarrow \frac{P_i}{P_j} > \frac{2^l}{2^r}$. **(3.6)**

We expect the distribution of the prime numbers given in **(3.1)** and **(3.2)** to produce the same results for the Triples.

Well, comparing the conditions **(3.5)** with Eqs. **(1.8)** and **(1.9)** we see:

$(r - 1)$ to be equivalent with $(2l - k - 1)$,

and

$(2l - r - 1)$ to be equivalent with $(k - 1)$.

Plug $r - 1 = 0$ into **(3.6)** we find

$\frac{P_i}{P_j} > \frac{2^l}{2} (= 2^{(l-1)})$. **(3.7)**

Now, plug $(2l - k - 1) = 0$ into

$2^{2l-k-1} P_i^2 > 2^{k-1} P_j^2 \rightarrow \frac{P_i}{P_j} > \frac{2^k}{2^l}$, **(1.11)**

and we have:

$\frac{P_i}{P_j} > \frac{2^{2l-1}}{2^l} (= 2^{l-1)})$,

equivalent to **(3.7)** .

Plug $2l - r - 1 = 0$ into **(3.6)** we find

$\frac{P_i}{P_j} > \frac{2^l}{2^{2l-1}} (= 1/2^{(l-1)})$. **(3.8)**

Now, plug $(k - 1 = 0)$ into

$2^{2l-k-1} P_i^2 > 2^{k-1} P_j^2 \rightarrow \frac{P_i}{P_j} > \frac{2^k}{2^l}$, **(1.11)**

then $\frac{P_i}{P_j} > \frac{1}{2^{l-1}}$

equivalent to **(3.8)**.

Hence, we find with the distribution of the prime numbers given in **(3.1)** and **(3.2)** the same

Triples indeed.
For creating the Pythagorean Triples you need an efficient algorithm for producing the prime numbers: *Generating Primes* , *Sieve of Atkin*, www.en.m.wikipedia.org .

Question: how many triples can we obtain for a given value of a ?
Well, for $a = 2^l$ and $\prod_s p_s^{n_s}$=1 , and for a particular value of $l \geq 2$ one triple is found. See *Table* at the end of section "A road map for generating Pythagorean triples". This is equivalent to the statement: for $n_s = 0$, one triple can be found.
Now we choose $a = 2^l p_t^{n_t}$, i.e. one odd prime number and $n_t \geq 1$. We find at least one triple. Since for $P_i = p_t^{n_t}$ and $P_j = 1$: $\frac{P_i}{P_j} > \frac{1}{2^{l-1}}$.
Is there another triple? We have two possibilities:
$k = 1$, then $P_i = 1$ and $P_j = p_t^{n_t}$,
and
$k = 2l - 1$, then $P_i = p_t^{n_t}$ and $P_j = 1$.
We analyse $k = 1$:
Plug into $\frac{P_i}{P_j} > \frac{1}{2^{l-1}}$ the values for P_i and P_j:
$l > 1 + \frac{n_t \ln(p_t)}{\ln 2}$.
Next $k = 2l - 1$:
$l < 1 + \frac{n_t \ln(p_t)}{\ln 2}$.
We conclude to find another triple for $l > 1 + \frac{n_t \ln(p_t)}{\ln 2}$ or $l < 1 + \frac{n_t \ln(p_t)}{\ln 2}$.
Hence: for $a = 2^l p_t^{n_t}$, i.e. one odd prime number and $n_t \geq 1$, we obtain two Pythagorean triples.
Conjecture: for a given a the number of triples equals 2^s,
where s is the number of odd-coprime-triples of which a is composed.
Note: we use the wording "co-prime", since for example 5^2 is a separate number in the triple of odd prime number series.

A final remark. You will notice not to find, e.g., a combination such as: (9, 12, 15). This is a non-primitive triple. Well, set $l = 2$, $P_i = P_j$, and $a = 12$ you will find this non-primitive solution . In this case by $P_i = P_j (\neq 1)$ a non-primitive solution has been forced. A simpler approach is by multiplying the primitive Triple (3, 4, 5) by any positive integer you like. The Triple (4, 3, 5) is found with $n_s = 0$ in **(1.4)** giving P_i and P_j equal 1 in **(1.8)** and **(1.9)** .