A Fractal Approach for Metabolism, other Scaling Constraints and a fat cat.

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Power laws.

For some time there is a discussion going on about the power law describing the relationship between the size of animals and their resting metabolism(www.physorg.com/news 184415311.html). In this article the $\frac{2}{3}$ law for metabolism is mentioned as the power law. This law follows from a straightforward dimensional analysis and the resting metabolic rates of warm-blooded animals.

In *Topics in Mathematical Modeling* by Tung a model of branching vascular networks is presented leading to a $\frac{3}{4}$ law. On the above mentioned website modelling of branching networks is applied by Dodds and a $\frac{2}{3}$ power law was found again by. So what should it be?

Now let us again apply straightforward dimensional analysis. However, we assume a more "fractal" like surface of warm-blooded animals. The larger animal has a dimension L and the smaller animal a dimension l. The volume(or mass) of the larger animal is n times larger than the smaller animal. The above $\frac{2}{3}$ law is found by the square of the ratio $\frac{L}{l} = n^{1/3}$

$\frac{L}{l} = n^{2/3}$.

We introduce now the fractal dimension for the surface of the animal. For this we choose the Hausdorff dimension 1.2619; the so called snowflake surface.

For the surface we have:

$(\frac{L}{l})^{1.2619}(\frac{L}{l})^{1.2619} = (\frac{L}{l})^{2.5238}$.

The volume ratio's of the larger and the smaller animal is

$\left(\frac{L}{l}\right)(\frac{L}{l})^{2.5238} = n,$

so $\frac{L}{l} = n^{1/3.5238}$.

The ratio of the "snowflake" surface is

$(\frac{L}{l})^{2.5238} = \left(n^{\frac{1}{3.5238}}\right)^{2.5238} \cong n^{0.72}$.

A power close to the $\frac{3}{4}$ power law.

Let us hope the biologists will find out for what circumstances either the $\frac{2}{3}$ law or the $\frac{3}{4}$ will apply.

Vesicles division.

In his book on "The Vital Question", Lane explains on the basis of surface-area-to-volume constrains why vesicles divide. Since new material has to be transported by the membrane enclosing the volume, there is a limitation in growth of the volume. In the meantime the surface is expanding so the vesicle has to divide. Doubling the membrane surface area, doubles the content of the membrane. Assuming a spherical geometry for the sake of illustration means more than doubling the volume. As we know this goes with a power of $\frac{3}{2}$. We could say that the information content of the vesicle is related to its surface and not to its volume. This something equivalent to the information of black holes. Well Susskind is nice reading on black holes, information and entropy.

A fat cat

This section is about surface areas and volumes.

As a starter let us look at the a given volume and compare the surfaces areas of o sphere and a cube bot6h with the same volume.

Well, this well-known.

We take the volume of the sphere with radius r as a base: $V = \frac{4}{3}\pi r^3$. The volume of the cube $V = h^3$, where h is the edge of the cube.

now we can express the edge h in terms of r : $h = r(\frac{4}{3}\pi)^{1/3}$. (1)

Now we determine the ratio of the surface area of the sphere to the surface area of the cube: $\frac{4\pi r^2}{6h^2}$. Substitute h given by (1) into $\frac{4\pi r^2}{6h^2}$ and we obtain for the ratio $(\frac{\pi}{6})^{1/3}$.

Hence the cube has a larger surface area than a sphere for a given volume.

Now for the fat cat. We again shall analyze surface areas for a given volume. The volume of the cat. A cat coils itself in order to take a nap. Why is that? Well, let us assume the body temperature of the cat to be higher than the temperature of the neighborhood. The loss of heat is proportional to the temperature difference with the neighborhood and the surface area of the cat. So, by coiling itself we can imagine the cat to make its surface area as small as possible. Can we prove such instinctive attitude?

Let us try a model. We assume the cat in active position to be represented by a cylinder with length L and radius r. The volume of the cat is: $V = \pi r^2 L$. The surface area of the active cat is: $S_a = 2\pi r L + 2\pi r^2$. (2)

We model the not so active cat to be coiled as a disc of radius R and thickness $2r$.

The volume of the cat in this position is: $V = 2\pi R^2 r$. The cat 's volume does not change. Consequently $V = \pi r^2 L = 2\pi R^2 r$ and we can express R in r and L.

Then we have $R = (\frac{rL}{2})^{1/2}$. (3)

The surface area of the inactive cat is $S_{ia} = 2\pi R^2 + 4\pi rR$. (4)

Now we compare both surface areas. The ratio of the surface area of the active cat to the surface area of the inactive cat is, using Eq. (3) and (4):

$$\frac{S_a}{S_{ia}} = \frac{2\pi rL+2\pi r^2}{2\pi R^2+4\pi rR}. \quad (5)$$

Substitute R given in (3) into (5) and we obtain:

$$\frac{S_a}{S_{ia}} = \frac{2(1+\frac{r}{L})}{1+(\frac{8r}{L})^{1/2}}. \quad (6)$$

By coiling the cat reduces heat transfer during the rest period. With $\frac{S_a}{S_{ia}} > 1$, we have $\frac{2(1+\frac{r}{L})}{1+(\frac{8r}{L})^{1/2}} > 1$. Solving this inequality gives: $\frac{r}{L} > \frac{1}{2}$.

Now for the fat cat. When does it no longer make sense to coil when taking a nap?

The condition is: $\frac{S_a}{S_{ia}} = 1$. With (6) we have $\frac{2(1+\frac{r}{L})}{1+(\frac{8r}{L})^{1/2}} = 1$.

Solving this quadratic equation, the condition is : $\frac{r}{L} = \frac{1}{2}$. As can be expected, the disc equals the cylinder: $2r = L$. Substitute $L = 2r$ into (3), we find $R = r$.

A fat cat indeed.

Literatuur.

Dodds, P., 2010, www.pjysorg.com/news184415311.html ;

Lane, N., *The Vital Question, why is life the way it is?,* Profile Books, 2015.

Susskind, L., *The Black Hole War*, Little, Brown and Company, 2008.

Tung, K. K., *Topics in Mathematical Modelling*, Princeton University Press, 2007.

Wikipedia, *A list of Hausdorff Dimensions.*